Completed

5.6 Notes: Graphing Rational Functions MAT133_Lawson

Many of the following examples come from College Algebra Handbook, College of the Ozarks,2019

Rational Functions:

A rational function is of the form $f(x) = \frac{p(x)}{q(x)}$, where p and q are polynomials, and $q(x) \neq 0$.

Since we typically will have variables in the denominator, this means that there will usually be values that have to be excluded from the domain of a rational function. (Recall that we must avoid all values for which the denominator equals zero.)

(Example 1) Find the domain of the rational functions

(a) $f(x) = \frac{-4x}{x^2-2x-24} = \frac{-4x}{(x-6)(x+4)}$

$x \neq 6, -4$

D: $(-\infty,-4) \cup (-4,6), (6,\infty)$

(b) $g(x) = \frac{2x^2-8}{x^2+2x} = \frac{2(x^2-4)}{x(x+2)} = \frac{2(x+2)(x-2)}{x(x+2)}$

$x \neq 0,\ x = -2$

D: $(-\infty,-2) \cup (-2,0) \cup (0,\infty)$

On the graph of a rational function, different behavior can occur as we approach a value that is excluded from the domain. To see an example of this, let's examine the graph of one of the most basic rational functions:

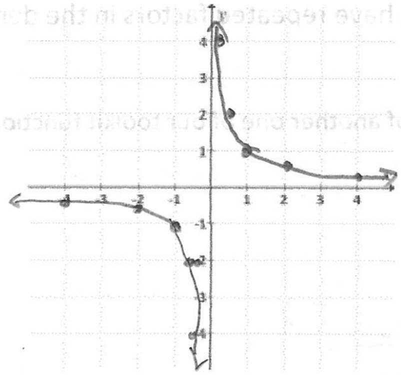
Graph $f(x) = \frac{1}{x}$

$x \neq 0$

Domain: $(-\infty,0) \cup (0,\infty)$

Range: $(-\infty,0) \cup (0,\infty)$

x	y
$-\frac{1}{4}$	-4
$-\frac{1}{2}$	-2
-1	-1
-2	$-\frac{1}{2}$
-4	$-\frac{1}{4}$



x	y
$\frac{1}{4}$	$\frac{1}{\frac{1}{4}} = 4$
$\frac{1}{2}$	2
1	1
2	$\frac{1}{2}$
4	$\frac{1}{4}$

As you can see with this function, the value of $x = 0$ that is excluded from the domain ends up being an asymptote for the graph of the function. Values excluded from the domain of a rational function will either be a vertical asymptote or a hole.

Vertical Asymptotes: A vertical asymptote (VA) is a line $x = a$ (← constant), which the graph approaches, but never crosses.

Note: we always write VA as an equation of a vertical line.

Holes: If $p(x)$ and $q(x)$ share a common factor, then the graph of $f(x) = \frac{p(x)}{q(x)}$ will have a hole at the x-value which corresponds to the common factor. (The y-coordinate of this hole can be found by substituting the x-value into the reduced form of the rational function.)

Review - Finding Intercepts: Recall that we find the x-intercepts of a function by solving $f(x) = 0$, and we find the y-intercepts by evaluating $f(0)$.

(Example 2) Find the vertical asymptotes and holes as well as the intercepts on the graph of the rational function.

(a) $f(x) = \dfrac{-4x}{x^2-2x-24} = \dfrac{-4x}{(x-6)(x+4)}$

VA: $x=6, x=-4$

holes: none

y-int: $f(0) = \frac{0}{-24} = 0 \rightarrow (0,0)$

x-int: $(0,0)$

(b) $g(x) = \dfrac{2x^2-8}{x^2+2x} = \dfrac{2\cancel{(x+2)}(x-2)}{x\cancel{(x+2)}} = \dfrac{2(x-2)}{x}$

VA: $x=0$

holes: $(-2,4)$ $\quad \frac{2(-4)}{-2} = \frac{-8}{-2}$

y-int: $g(0) = \frac{-8}{0} \rightarrow$ undefined $\rightarrow$ none

x-int: $(2,0)$

When we have repeated factors of polynomial, we call the repeated zeros **multipliticities,** and our graph "bounces off" the x-axis at these zeros if they are real and have multiplicity that is even. This is still true for rational functions, that is, if we have repeated factors in the numerator.

What happens when we have repeated factors in the denominator? How does this affect a graphs behavior around an asymptote?

Let's examine the graph of another one of our toolkit functions and compare it to our previous example.

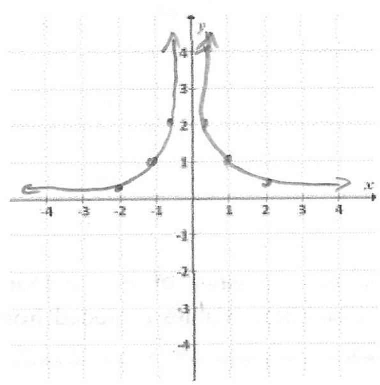
Graph $f(x) = \dfrac{1}{x^2}$

$x \neq 0$

Domain: $(-\infty, 0) \cup (0, \infty)$

$y > 0$

Range: $(0, \infty)$



x	y
-2	$\frac{1}{4}$
-1	1
$-\frac{1}{2}$	4
$\frac{1}{2}$	4
1	1
2	$\frac{1}{4}$

End Behavior of Rational Functions:

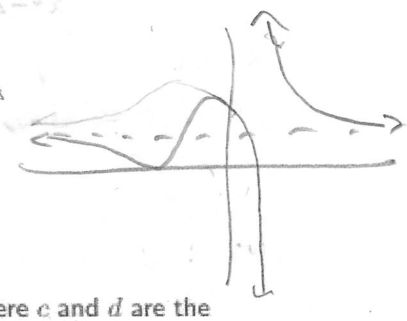
constant

A horizontal asymptote (HA) is a line $y = b$, that the graph approaches as $x \to \pm\infty$. (The graph can cross a HA, but doesn't have to.) In other words, the asymptotes describe the end behavior of the graph, as $x \to \pm\infty$.

For a rational function

$$f(x) = \frac{p(x)}{q(x)} = \frac{cx^n + \dots}{dx^m + \dots}$$

HA



where n and m are the degrees of $p(x)$ and $q(x)$, respectively:

- If $n < m$, then $y = 0$ is a horizontal asymptote of the function f.
- If $n > m$, then f has NO horizontal asymptotes.
- If $n = m$, then $y = \frac{c}{d}$ is a horizontal asymptote of the function f, where c and d are the leading coefficients of $p(x)$ and $q(x)$, respectively.

If there is no horizontal asymptote, the end behavior is just that of the simplified leading term: $\frac{cx^n}{dx^m}$

Consider the end behavior of each.

(eg) $f(x) = \frac{5x^2 + 3x + 1}{2x - 1}$, $\quad g(x) = \frac{2x - 1}{5x^2 + 3x + 1}$, $\quad h(x) = \frac{2x^2 - 1}{5x^2 + 3x + 1}$

no HA $\qquad y = 0 \qquad y = \frac{2}{5}$

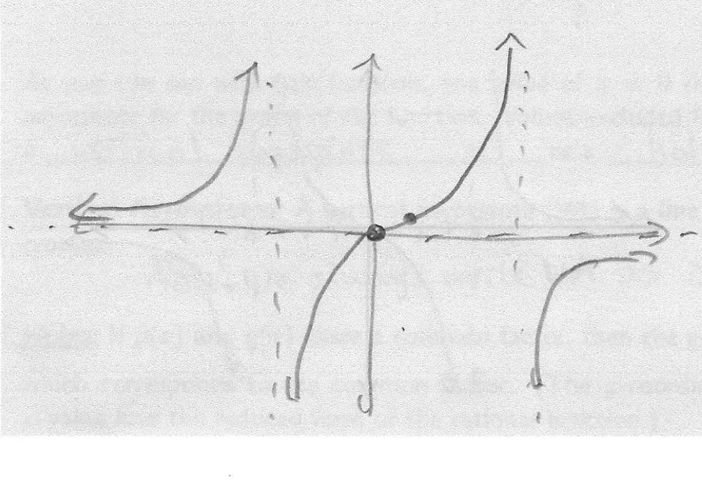
$\frac{5x^2}{2x} = \frac{5x}{2}$ end behavior: ↗ ↙

(Example 3) Find the end behavior/horizontal asymptotes of the graph of the rational function.

(a) $f(x) = \frac{-4x}{x^2 - 2x - 2} = \frac{-4}{(x-6)(x+4)}$

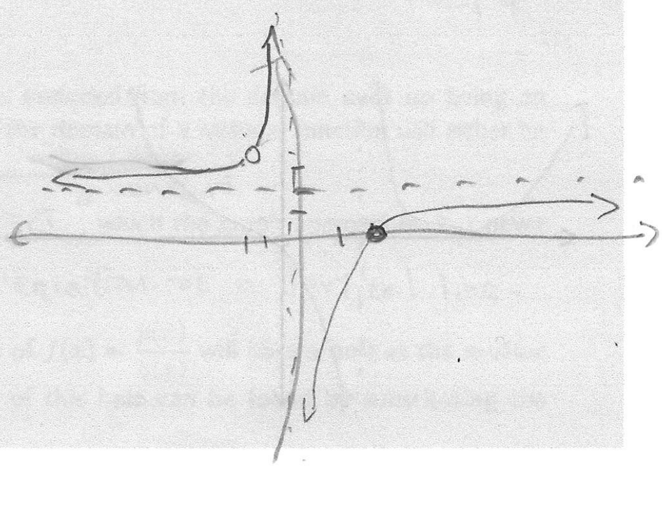
HA $y = 0$

$f(1) = \frac{-4}{(-5)(5)} = \frac{-4}{-25}$



(b) $g(x) = \frac{2x^2 - 8}{x^2 + 2x} = \frac{2(x-2)}{x}$

HA, $y = 2$

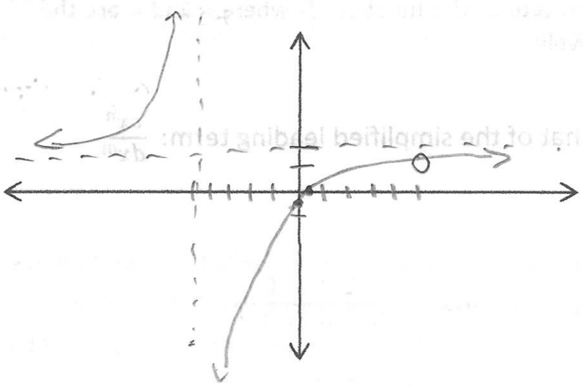


(Example 4) Putting together all VA, HA, and intercepts, sketch the graph of each rational function. It may help to plot additional points around your x-intercepts and VA as needed to verify whether your graph is positive or negative.

$2x^2-10x-x+5$

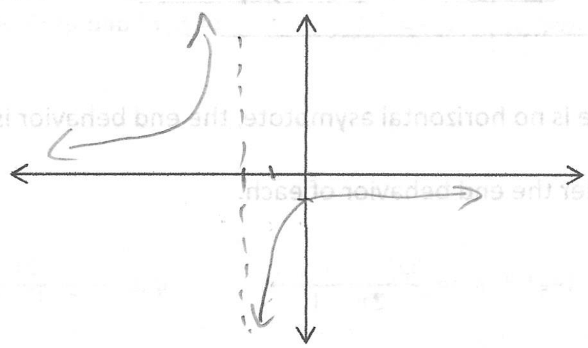
(a) $f(x)=\frac{2x^2-11x+5}{x^2-25} = \frac{(2x-1)(x-5)}{(x+5)(x-5)}$

x-int: $(\frac{1}{2}, 0)$
y-int: $(0, -\frac{1}{5})$
VA: $x=-5$
holes $(5, \frac{9}{10})$
HA: $y=2$



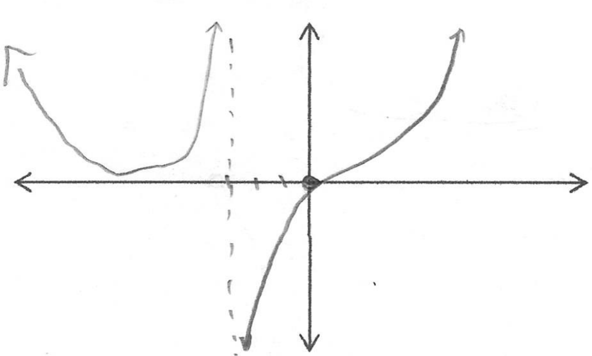
(b) $g(x)=\frac{-4}{x^2+4x+4} = \frac{-4}{(x+2)^2}$

x-int: none
y-int: $(0,-1)$
VA: $x=-2$
holes: None
H.A. $y=0$



(c) $h(x)=\frac{3x^3}{x+3}$

x-int: $(0, 0)$
y-int ↗
VA $x=-3$
holes: none
end behavior: $\frac{3x^3}{x} = 3x^2$ ↑ ↗



(d) $k(x)=\frac{5x}{16-x^2} = \frac{5x}{(4-x)(4+x)}$

x-int: $(0,0)$
y-int ↗
VA $x=4, x=-4$
holes: none
HA: $y=0$
$K(1)=\frac{5}{15}=\frac{1}{3}$